Geometry of four-manifolds

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Abstract

In these notes, I will report on my study about some classical results from Yang-Mills theory and four-dimensional geometry. These results are due to several great mathematicians, namely Donaldson, Uhlenbeck, Atiyah and so on. I will first explain basic concepts in Yang-Mills theory, then turn to instantons and ADHM construction. Finally we see how these theories can be applied to the study of four-dimensional topology.

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1 Preface

These notes were obtained while the author was learning the theory of 4-dimensional manifolds. The main reference is none other than the book *The Geometry of Four-Manifolds* written by Donaldson and Kronheimer, which applies the geometric aspect of Yang-Mills theory to the study of 4-manifolds and thus of particular interests.

2 Yang-Mills theory

This section explains mathematical aspect of Yang-Mills theory.

Yang-Mills theory is a generalization of Maxwell's theory on electromagnetism, used to describe the weak force and the strong force in subatomic particles. It was introduced by physicists Yang C.N. and Robert L.Mills. Surprisingly, Simons and Yang discovered the correspondences between Yang-Mills theory and fiber bundle theory: gauge potential to connection on a principal bundle, gauge field to curvature, electromagnetism to connections on $U(1)$-bundle, Dirac's monopole quantization to classfication of $U(1)$-bundle and so on. For a reference, see [7].

We first review some basic concepts about connections and curvature on vector bundles. Then we will give two heuristic theorems, both of which illustrate the principle that curvature reflects the property of connections.

2.1 Connections and curvature

Given a vector bundle E(complex or real) over X, a connection on E is a map:

$$\nabla_A : \Omega^0_X(E) \to \Omega^1_X(E)$$

which satisfies the Leibnitz rule:

$$\nabla_A(fs) = df \otimes s + f\nabla_A s$$

Here Ω^p_X denotes sections of $\bigwedge^p T^*X \otimes E$, i.e. p-forms with value in E.

Provided with a trivialization τ of E, a connection ∇_A can be written as:

$$\nabla_A = d + A^\tau$$

where d is the exterior differential operator of X and A^τ is a matrix of 1-forms. Use the coordinate represention of sections, it acts on $s = (f_1, \ldots, f_n)$ as follows:

$$\nabla_A s = d \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} + A^\tau \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

If the base neighborhood has coordinate chart $x = (x_1, \ldots, x_m)$, we also denots its components by $\nabla_i = \frac{\partial}{\partial x^i} + A^\tau_i$ where A^τ_i is a matrix of functions and $\nabla_A = \nabla_i \otimes dx^i$.

If the bundle E admits a metric, we will sometimes require the connection compatible with the metric.

Definition 2.1. *We say $u : E \to E$ a gauge transformation if u is a vector bundle automorphism covering $\pi : E \to X$ and preserve metric if E has. All gauge transformations form a group called the gauge group of E. A gauge transformation u acts on a conncetion ∇_A as follows: $\nabla_{u(A)}s = u\nabla_A(u^{-1}s)$.*

A basic fact is that we can always extend a connection to a differential operator $d_A : \Omega^p_X(E) \to \Omega^{p+1}_X(E)$ uniquely characterized by:

- $d_A = \nabla_A$ on $\Omega^0_X(E)$
- $d_A(\omega \otimes\ s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla s$, for any $\omega \in \bigwedge^p T^*X$ and $s \in \Omega^0_X(E)$

where $\omega \wedge \nabla s$ means wedge product in coefficients.

The curvature of ∇_A is defined to be $F_A = d_A^2$ on $\Omega^0_X(E)$. By Leibnitz rule, F_A acts tensorially on sections of E and hence belongs to $\Omega^2_X(End(E))$.

Locally, we can write curvature form as:

$$F^\tau_A = dA^\tau + A^\tau \wedge A^\tau$$

This is called the structural equation. If we denote curvature by its components:$F = \sum_{i<j} F_{ij} dx^i \wedge dx^j$, where F_{ij} is a matrix of functions, then the structural equation becomes:

$$F_{ij} = [\nabla_i, \nabla_j] = \frac{\partial A^\tau_j}{\partial x_i} - \frac{\partial A^\tau_i}{\partial x_j} + \left[A^\tau_i, A^\tau_j\right]$$

We see from the above that for a line bundle, the non-linear term vanishes.

Now suppose E is a Hermitian bundle over a Riemannian manifold M, we can define the Yang-Mills functional of a given connection ∇_A as follows:

$$\|F_A\|^2 = \int_M |F_A|^2 d\mu$$

And it's easy to obtain its Euler-Lagrange equation:

$$d^*_A F_A = 0$$

which is known as Yang-Mills equation. When structure group is the abelian group $U(1)$, this equation is just classical Maxwell equation.

2.2 Integrability: flat connection

A general principle is that curvature reflects the information of connection. We begin with a fundamental integrability theorem for connections.

Theorem 2.1. *If E is a bundle with metric over hypercube $H = \{x \in \mathbb{R}^d : |x_i| < 1, i = 1, ..., d\}$ and A a flat connection on E. Then there is a bundle isomorphism preserving metric taking E to the trivial bundle over H and A to the product connection.*

The proof is standard in geometry. We fix a basis in the fiber over the origin and transport them parallelly to different fibers. The flat condition guarantees that parallel transportation does not depend on the choice of path and thus gives a well-defined local frame and hence a trivialization of E.

Then we turn to another kind of integrable theorem which concerns about holomorphic bundles. Suppose we have a holomorphic structure $\mathscr{E}$ on a complex bundle E over a complex manifold Z. There is a canonical differential operator $\bar{\partial}_{\mathscr{E}} : \Omega_Z^{0,q}(E) \to \Omega_Z^{0,q+1}(E)$ uniquely determined by the following properties:

1. $\bar{\partial}_{\mathscr{E}}(fs) = \bar{\partial} f \otimes s + f\bar{\partial}_{\mathscr{E}} s$;
2. $\bar{\partial}_{\mathscr{E}} s$ vanishes on an open subset $U \subset Z$ if and only if s is a holomorphic section over U.

We construct it explictly. Given a holomorphic trivialization, define $\bar{\partial}_{\mathscr{E}}$ on $\Omega_Z^0(E)$ by ordinary $\bar{\partial}$-operator then extends it to $\Omega_Z^{0,q}(E)$. This is well-defined since $\bar{\partial}$-operator acts trivially on holomorphic transition functions. Obviously this operator satisfies that $\bar{\partial}_{\mathscr{E}}^2 = 0$.

Conversely, consider a "partial connection" $\bar{\partial}_\alpha : \Omega_Z^0(E) \to \Omega_Z^{0,1}(E)$ on a smooth complex bundle E over a complex manifold Z which satisfies Leibnitz rule. This can be obtained in particular by a connection A on E: $d_A = \partial_A \oplus \bar{\partial}_A : \Omega_Z^0(E) \to \Omega_Z^{1,0}(E) \oplus \Omega_Z^{0,1}(E)$. We want to ask if a partial connection is induced by a holomorphic structure, which leads to the following definition.

Definition 2.2. *We say a partial connection $\bar{\partial}_\alpha$ is integrable if for any $z \in Z$, there exists a trivialization τ of E such that $\alpha^\tau = 0$ where $\bar{\partial}_\alpha = \bar{\partial} + \alpha^\tau$ is the local coordinate representaion of a connection.*

FACT: If $\bar{\partial}_\alpha$ is integrable, then it is induced by a holomorphic structure. This is because the transition function of any two such trivializations is holomorphic.

Now we can state our integrability theorem.

Theorem 2.2. *A partial connection $\bar{\partial}_\alpha$ on a smooth complex bundle E over a complex manifold Z is integrable if and only if $\bar{\partial}_\alpha^2 = 0$.*

Proof. Since we only need to prove it locally, we may assume that Z is a polydisc $K(1) = z \in \mathbb{C}^d : |z_j| < 1, j = 1, ..., d$. Our assumption is that $\bar{\partial}\alpha + \alpha \wedge \alpha = 0$, and we want to show that there is a smaller polydisc $K(r) = \{z \in \mathbb{C}^d : |z_j| < r, j = 1, ..., d\}$ and a complex gauge transformation $g : K(r) \to Gl(n; \mathbb{C})$ satisfying $g\alpha g^{-1} - (\bar{\partial} g)g^{-1} = 0$.

We prove by induction. For $n = 1$, the integrable condition is vacuous. We write $\alpha = \rho d\bar{z}$ where ρ is a matrix of functions on $D(1) \subset \mathbb{C}$. We are going to solve:

$$\frac{\partial g}{\partial \bar{z}} - g\rho = 0$$

where g is invertible.

Still, since it is only a local problem, we use a cut-off function to assume that ρ has compact support near the origin. In particular, ρ is defined on $\mathbb{C}$ and is bounded. Consider $\delta_r : \mathbb{C} \to \mathbb{C}, z \mapsto rz$, the equation becomes:

$$\frac{\partial g}{\partial \bar{z}} = r\rho(rz)g(z)$$

Therefore we can assume that $N = \sup_{z \in \mathbb{C}} |\rho|$ is as small as we want. Let $g = 1 + f$, the equation becomes:

$$\frac{\partial f}{\partial \bar{z}} = (1 + f)\rho$$

Now consider the Cauchy kernel $-\frac{1}{2\pi i}z$ and the operator:

$$(L\theta)(w) = -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\theta(z)}{z - w} dz \wedge d\bar{z}$$

for compactly supported function θ. It is a standard fact that:

$$\frac{\partial}{\partial \bar{z}}(L\theta) = \theta$$

So we can transfer the PDE into:

$$f = L(\rho + f\rho)$$

If we can solve this equation and get solution f such that $\|f\|_\infty$ is small, then $g = 1 + f$ will be invertible. Also, elliptic regularity for $\bar{\partial}$-operator will imple that any bounded solution is actually smooth.

By basic coordinate change, we can show that for any unit disc D in $\mathbb{C}$, we have:

$$\int_D \frac{1}{|z|} dz \wedge d\bar{z} \leq \int_0^1 2\pi r \frac{1}{r} dr = 2\pi$$

Because $\text{supp}\rho \subset D(1)$, for any h we have:

$$\|L(h\rho)\|_\infty \leq N\|h\|_\infty$$

If $N < 1$, $T(\varphi) = L(\rho + \varphi\rho)$ is a contraction mapping from L^∞ to itself, and thus have a unique fixed point f. Also, since we have estimate:

$$\|f\|_\infty = \|L((1 + f)\rho)\| \leq N\|1 + f\|_\infty \leq N(1 + \|f\|_\infty)$$

if we assume $N < \frac{1}{2}$, then:

$$\|f\|_\infty \leq N + \frac{1}{2}\|f\|_\infty$$

which implies that $\|f\|_\infty \leq 2N$. Therefore we can make $(1 + f)$ invertible by choosing N small.

For general case, we need the following fact concerning about solutions' dependence on parameters:

FACT: If $\rho = \rho(z; \xi, \eta)$ is holomorphic in ξ and smooth in η, then solution $g = g(z; \xi, \eta)$ is holomorphic in ξ and smooth in η.

Now let $\alpha = \sum \alpha_j d\bar{z}_j$ and $\alpha_j = 0$ for $j = 1, ..., p$. Note that this condition is preserved by an automorphism g which are holomorphic in $z_1, ..., z_p$. Take $\rho = \alpha_{p+1}$, then by flat condition:

$$0 = \Phi_{j,p+1} = \frac{\partial \alpha_{p+1}}{\partial \bar{z}_j} - \frac{\alpha_j}{\partial \bar{z}_{p+1}} + [\alpha_j, \alpha_{p+1}] = \frac{\partial \alpha_{p+1}}{\partial \bar{z}_j}$$

for $j = 1, ..., p$. Thus ρ is holomorphic in $z_1, ..., z_p$. Like before, we solve the equation:

$$\frac{\partial h}{\partial \bar{z}_{p+1}} = h\rho$$

with $z_j, j \neq p+1$ as parameters. The solution is holomorphic in $z_1, ..., z_p$ and hence gives a trivialization which preserves the condition that $\alpha_j = 0, j = 1, ..., p$. Also by construction $\alpha_{p+1} = 0$. By induction our conclusion holds. $\square$

2.3 Uhlenbeck's theorem

A flat connection can be locally represented by a zero connection matrix. It is natural to ask if connection with small curvature can be represented by a "small" connection matrix. This is what Uhlenbeck theorem tells us.

Let B^4 be the unit ball in $\mathbb{R}^4$ and $m : \mathbb{R}^2 \to S^4$ be the standard stereographic map which maps the unit ball to a hemisphere in S^4. We fix the metric on B^4 as the pullback of the round metric on S^4 by m. This is conformal to the flat metric, so in this particular dimension four the L^2-norm of 2-forms is the same for both metrics. This choice of metric is merely a convenience, as though what indeed is done originally by Uhlenbeck is the standard metric.

We work with trivial Hermitian bundles E over B^4 and conpatible $U(n)$-connections. We denote by A_r the radial component $\sum(\frac{x_i}{r})A_i$ of the connection matrix and $\|\cdot\|_{L^p_k}$ the Sobolev norm for sections with k-th weak derivatives and all of these derivatives are in L^p space.

Theorem 2.3 (Uhlenbeck)**.** *There are constants $\epsilon_1, M > 0$ such that any connection A on the trivial bundle over $\bar{B}^4$ with $\|F_A\|_{L^2} < \epsilon_1$ is gauge equivalent to a connection $\tilde{A}$ over $\bar{B}^4$ with*

1. $d^*\tilde{A} = 0$;
2. $\lim_{|x| \to 1} \tilde{A}_r = 0$;
3. $\|\tilde{A}\|_{L^2_1} \leq M \|F_{\tilde{A}}\|_{L^2}$.

Moreover, for suitable constant ϵ_1, M, the connection $\tilde{A}$ is uniquely determined by these properties, up to a constant transformation $u_0 \in U(n)$.

We provide basic steps of the proof which are of interest in geometry, but omitt some technical details. The proof is different from the original proof of Uhlenbeck but essencially the same. For a reference, see [3].

First we will work on S^4 and then turn to $\bar{B}^4$, which helps us to deal with boundary condition. Also, some topological property of S^4 helps us to make some estimate.

By a one-parameter family of connections we mean a continuous family of connection matrices defined over $S^4 \times \mathbb{R}$, both smooth in S^4 variable and all partial derivatives are jointly continuous in two variables.

Proposition 2.1. *There is a constant $\zeta > 0$ such that if $B'_t, t \in [0,1]$ is a one parameter family of connections on the trivial bundle over S^4 with $\|F'_B\| < \zeta$ with B'_0 trivial, then for each t there exists a gauge transformation u_t such that $u_t(B'_t) = B_t$ satisfies:*

1. $d^* B_t = 0$;
2. $\|B_t\|_{L^2_1} < 2N\|F_{B_t}\|$ *if* $B_t \neq 0$

where N is a universal constant which is given by a lemma below.

Let S be the set of all $t \in [0,1]$ such that the proposition holds. We prove this by continuity method.

To prove this, we need two lemmas for which we omit their proof.

lemma 2.1 (Key lemma)**.** *There exists constant $N, \eta > 0$ such that for any connection B on the trivial bundle over S^4 with $d^*B = 0$, if $\|B\|_{L^4} < \eta$ then $\|B\|_{L^2_1} \leq N\|F_B\|_{L^2}$.*

We point out here that the proof of this lemma depends on the topological proterty of S^4.

lemma 2.2 (Estimate of higher derivatives)**.** *There is a constant $\eta' > 0$ such that if the connection matrix B in previous lemma has $\|B\|_{L^2} < \eta'$ then for each $l \geq 1$, we have:*

$$\|B\|_{L^2_{l+1}} \leq f_l(Q_l(B))$$

where f_l is a universal continuous function with $f_l(0) = 0$ and $Q_l(B) = \|F_B\|_{L^\infty} + \sum_0^l \|\nabla_B^i F_B)\|_{L^2}$.

2.3.1 Method of continuity: closedness

Now we prove the closedness of proposition 2.1. The main step is due to following observation:

Observation: Assume $\{A_n\}$,$\{B_n\}$ are unitary connections on the trivial bundle over S^4 which are gauge equivalent. If $\{A_n\}$ and $\{B_n\}$ converge in C^∞ to A_∞ and B_∞, i.e. the original sequence and every derivative sequence converges uniformly on every compact subset. Then A_∞ and B_∞ are gauge equivalent.

Proof. Let $\{u_n\}$ be the gauge transformation sequences that take $\{A_n\}$ to $\{B_n\}$. Rewrite transformation formula as:

$$du_n = u_n A_n - B_n u_n$$

From this formula, we see that if $\{u_n\}$ is bounded in C^r, so is $\{du_n\}$ since $\{A_n\}$ and $\{B_n\}$ are bounded in C^∞. For the case $r = 0$, $\sup_{n\geq 1} \|u_n\|_\infty \leq \sup_{m\in U(n)} \|m\|_\infty < \infty$ since $U(n)$ is compact. Therefore by induction we see that $U(n)$ is bounded in C^∞.

By Arzela-Ascoli theorem and diagonal argument, we can find a subsequence that converges to u_∞ in C^∞. We assume it is the full sequence. Take limit we obtain:

$$B_\infty = u_\infty A_\infty u_\infty^{-1} - du_\infty u_\infty^{-1}$$

which shows that A_∞ and B_∞ are gauge equivalent. □

Remarks:

1. The result is false for connections with non-compact gauge groups.
2. The proof actually adapts to any unitary bundle over any manifold.
3. If $\{A_n\}$ and $\{B_n\}$ are bounded in C^∞, by taking subsequences we can apply the above argument.

Proof. Take $\zeta < \min\{\frac{\eta}{2CN}, \frac{\eta'}{2CN}\}$ where η and η' is given by previous two lemmas and C is the Sobolev constant. We may assume that $C \geq 1$. If $t \in S$, then by Sobolev embedding we have:

$$\|B_t\|_{L^4} \leq C\|B_t\|_{L_1^2} < 2NC\|F_{B_t}\|_{L^2} < 2NC\zeta$$

Apply key lemma:

$$\|B_t\|_{L_1^2} \leq N\|F_{B_t}\|_{L^2}$$

This is, we have gone from an open condition $\|B_t\|_{L_1^2} < 2N\|F_{B_t}\|_{L^2}$ to a stronger closed result $\|B_t\|_{L_1^2} \leq N\|F_{B_t}\|_{L^2}$.

On the other hand, to apply estimate of higher derivatives, we observe that:

$$\|B_t\|_{L^2} \leq \|B_t\|_{L_1^2} < 2N\|F_{B_t}\|_{L^2} < 2N\xi < \frac{\eta'}{C} \leq \eta'$$

since we have assumed that $C \geq 1$.

By gauge invariance of covariant derivative, we find that:

$$\|\nabla_{B_t}^{(j)} F_{B_t}\|_\infty = \|\nabla_{B_t'}^{(j)} F_{B_t'}\|_\infty \leq \sup_{t\in[0,1], x\in S^4} |\nabla_{B_t'}^{(j)} F_{B_t'}| < \infty$$

Here we use that B_t' is a one-parameter familiy of connections and the compactness of $S^4 \times [0,1]$.

Therefore, for any $l \geq 1$ we obtain a uniform L_{l+1}^2-bound on all derivatives of B_t for any $t \in S$. Then by Sobolev inequality, we obtain uniform L_∞-bounds since the base is compact.

Now we can apply previous argument to $\{B_t\}_{t\in S}$. For any sequence $\{t_n\}$ converges to $t_0 \in [0,1]$, by taking subsequence we can conclude that B_{t_n} converges in C^∞ to B_{t_0}, which is gauge equivalent to B'_{t_0}. The condition is preserved since we actually have a closed condition $\|B_{t_0}\|_{L_1^2} \le N\|F_{B_{t_0}}\|_{L^2}$. Thus $t_0 \in S$ and S is closed. □

2.3.2 Method of continuity: openness

In this part we sketch the proof of openness since it is more like a PDE argument.

The gauge fixing equation is written as:

$$d^*(u_t(B'_t)) = d^*(u_t B'_t u_t^{-1} - (du_t)u'_t) = 0$$

Let $t_0 \in S$, we may assume that $B_{t_0} = B'_{t_0} = B$. Put $B'_{t_0+\delta} = B + b_\delta$, we seek for a solution $u_{t_0+\delta}$ in the form:

$$u_{t_0+\delta} = \exp(\chi_\delta)$$

Let $H(\chi, b) = d^*(e^\chi(B+b)e^{-\chi} - d(e^\chi)e^{-\chi})$, then the equation becomes:

$$H(\chi_\delta, b_\delta) = 0$$

This defines a smooth map between Banach spaces, then we use implicit function theorem on Banach space to conclude the openness.

2.3.3 Completion of the proof

We finally come to the proof of Uhlenbeck theorem. We need two steps to come from S^4 to $\bar{B}^4$.

First, there is a canonical path from any connection A to the product connection:

$$\delta_t : \mathbb{R}^4 \to \mathbb{R}^4, x \mapsto tx$$

for $t \in [0,1]$. Let $A_t = \delta_t^*(A|_{\bar{B}(0,t)})$, then A_t is a familiy of connections on the trivial bundle over $\bar{B}^4$. Clearly $A_0 = 0$,$A_1 = A$.

By conformal invariance in dimension 4, we have:

$$\int_{\bar{B}^4} |F_{A_t}|_g^2 d\mu = \int_{\bar{B}^4} |F_{A_t}|^2_{\delta_t^* g} d\mu_{\delta_t^* g} = \int_{\bar{B}(0,t)} |F_A|^2 d\mu \le \int_{\bar{B}^4} |F_A|^2 d\mu$$

Therefore the L^2-norm of F_{A_t} can be controlled uniformly by that of A.

Second, we define a mapping $p : S^4 \to \bar{B}^4$ as follows: it projects lower-hemisphere directly to $\bar{B}^4$ and takes upper-hemisphere first to the lower-hemisphere by reflection then projects to $\bar{B}^4$. For a connection matrix α over $\bar{B}^4$, let $\beta = p^*(\alpha)$,$F_\beta = p^*(F_\alpha)$. Clearly we have:

$$\int_{S^4} |F_\beta|^2 d\mu = 2\int_{\bar{B}^4} |F_\alpha|^2 d\mu$$

Therefore we can decuce main assertion of Uhlenbeck theorem by our established result on S^4, since the curvature is bounded by a factor of $\sqrt{2}$. The only possible obstruction is that p is not smooth on S^3. But p and dp can be approximated uniformly by smooth maps since p is Lipschitz. Thus the gauge transformation can be obtained by approximation. Boundary condition can also be satisfied by approximation process. Finally, uniqueness asssertion can be deduced by controlling of L_1^2-norm.

3 Instantons and ADHM construction

We explain the concept of instantons in this section.

Given a Riemannian four-manifold M, Hodge$*$ operator splits Ω^2_M into $\Omega^+_M \oplus \Omega^-_M$, the self-dual parts and anti-self-dual parts. It can be extended to any bundle-valued 2-forms and in particular the curvature tensor F_A splits into $F_A^+ \oplus F_A^-$. This leads to the following definition:

Definition 3.1. *We say a connection A is anti-self-dual if $F_A^+ = 0$, which is abbreviated as ASD connections. ASD connections are also called instantons from physical view of point.*

By Bianchi identity, ASD connections trivially satisfy Yang-Mills equation.

3.1 ASD condition: two interpretations

3.1.1 Topological interpretation

We first explain a topological interpretation of instantons.

Given a complex vector bundle E over M, Chern-Weil theory tells us that Chern class can be given by any connection A as follows:

$$c_1(E) = \frac{i}{2\pi}[Tr(F_A)]$$

$$c_2(E) = \frac{1}{2}c_1(E)^2 + \frac{1}{8\pi^2}[Tr(F_A \wedge F_A)]$$

For our case, these two classes are satisfactory since we have:

Theorem 3.1. *The first Chern class $c_1 \in H^2(M;\mathbb{Z})$ classfies $U(1)$-bundles over any CW complex M. The second Chern class $c_2 \in H^2(M;\mathbb{Z})$ classifies $SU(2)$-bundles over any compact oriented four-manifold M^4.*

For a proof, see appendix in [8].

Now for $SU(2)$-bundles, $c_1(E)$ vanishes and hence $c_2(E) = \frac{1}{8\pi^2}[Tr(F_A \wedge F_A)]$. This is also equivalent to consider the second Chern number:

$$c_2 = \int_M Tr(F_A \wedge F_A) \in \mathbb{Z}$$

If the base space is an oriented Riemannian manifold, we then have:

$$Tr(F_A^2) = -(|F_A^+|^2 - |F_A^-|^2)d\mu$$

where $d\mu$ is the oriented volume form. Therefore second Chern number can be given by:

$$c_2 = \frac{1}{8\pi^2}(\|F_A^-\|^2 - \|F_A^+\|^2)$$

This gives us a lower bound of Yang-Mills functional on the space of connections:

$$\|F_A\|^2 = \|F_A^-\|^2 + \|F_A^+\|^2 \geq 8\pi^2|c_2|$$

When $c_2 \geq 0$, this bound is obtained precisely when A is an ASD connection.

3.1.2 Complex geometric interpretation

The second interpretation of ASD condition is related to complex manifold and holomorphic structures.

Recall we have discussed partial connections on complex bundles and integrability theorem. From this we see that given a connection A, its partial connection $\bar{\partial}_A$ is integrable if and only if its (0,2)-component vanishes, i.e. $F_A^{0,2} = 0$.

Now we introduce Hermitian metric on bundles. This gives us a canonical way to identify partial connections and unitary connections.

lemma 3.1. *If E is a complex vector bundle over a complex manifold Z with a Hermitian metric on the fibers, then for each partial connection $\bar{\partial}_\alpha$ on E, there is a unique unitary connection A such that $\bar{\partial}_A = \bar{\partial}_\alpha$.*

For Hermitian bundle and unitary connection, an integrable condition can be stated as:

Proposition 3.1. *A unitary connection on a Hermitian complex vector bundle over Z is compatible with a holomorphic structure if and only if it have curvature of type (1,1) and in this case the connection is uniquely determined by the metric and holomorphic structure.*

This canonical connection is called Chern connection.

We focus on the case of a complex surface with a Hermitian metric on its tangent bundle. The complex structure and metric together defines a (1,1)-form ω, so we have a pointwise orthogonal decomposition of $\Omega^{1,1}$:

$$\Omega^{1,1} = \Omega_0^{1,1} \oplus \langle\omega\rangle$$

Algebraic caculation shows that:

lemma 3.2. *The complexified self-dual forms are*

$$\Omega^+ = \Omega^{2,0} \oplus \langle\omega\rangle \oplus \Omega^{0,2}$$

and the complexified anti-self-dual forms are

$$\Omega^- = \Omega_0^{1,1}$$

Now we bring all the above together. For any connection A on E, put $\widehat{F}_A = (F_A, \omega)$ be the (1,1)-component of the curvature along the metric form. Then we have:

Proposition 3.2. *If A is an ASD connection on a complex bundle E over a Hermitian complex surface Z, the the operator $\bar{\partial}_A$ defines a holomorphic structure on E. Conversely, if $\mathscr{E}$ is a holomorphic structure on E, and A is a compatible unitary connection, then A is ASD if and only if $\widehat{F}_A = 0$.*

3.2 ADHM construction

In this section we discuss about instantons over S^4. It was first observed by Atiyah and Ward(see [10]) that the holomorphic structure of vector bundles on $\mathbb{C}P^3$ with certain real and symplectic structures can be used to describe ASD connections on S^4. Later with help of index theorem, Atiyah, Hitchin and Singer were able to show that all instantons on S^4 moduli the gauge equivalence are a smooth manifold of dimension $8k-3$ where k is the Pontrjagin index(see [11]). Following this, Atiyah, Drinfeld, Hitchin and Manin constructed these instantons via just linear algebra(see [12]). Their construction is known as ADHM construction.

We point our here that we will use the conformal invariance of dimension four, going back and forth between $\mathbb{R}^4$ and S^4 frequently to prove their construction.

3.2.1 Riview of spin structure

First we review some facts about spin structure.

Let S be a two-dimensional complex vector space with a Hermitian metric and a compatible complex symplectic form $\lambda \in \bigwedge^2_{\mathbb{C}} S^*$ with $|\lambda| = 2$. This defines an anti-linear map $J : S \to S$ by: $\langle x, Jy \rangle = \lambda(x, y)$. Then $J^2 = -1$ and makes S into a one-dimensional quaternionic vector space. Let S^+,S^- be a pair of such spaces, we consider the space $\mathrm{Hom}_J(S^+, S^-)$ of complex linear maps interwine the J actions. This is a four-dimensional real vector space. It also carries an Euclidean metric, normalized such that the unit vertors are precisely the map preserving both Hermitian metrics and the symplectic forms.

Now given a four-dimensional Euclidean space V, we say a spin structure on V is a pair of complex vector spaces S^+,S^- as above and an isomorphism $\gamma : V \to \mathrm{Hom}_J(S^+, S^-)$ compatible with the Euclideam metric. In the standard Euclidean space $\mathbb{R}^4$, we can take:

$$\gamma(e_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma(e_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \gamma(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

where $\{e_i\}$ are natural basis.

Given a spin structure on V, consider the compositiong $\gamma^*(e)\gamma(e')$, which is an endomorphism of S^+. Here $\gamma^*(e)$ is the adjoint with respect to the Hermitian metric. It satisfies the following relation

1. $\gamma^*(e)\gamma(e) = 1$, if e is a unit vector,
2. $\gamma^*(e)\gamma(e') + \gamma^*(e')\gamma(e) = 0$, if e and e' are orthogonal.

This means that $\wedge^2(V)$ acts on S^+ by:

$$(e \wedge e')s = -\gamma^*(e)\gamma(e')s$$

where e and e' are orthogonal. Moreover $\wedge^-$ acts trivially here and we get a natural isomorphism:

$$\rho : \wedge^+ \to \mathfrak{su}(S^+)$$

where the right hand denotes the trace-free, skew-Hermitian endomorphisms.

3.2.2 Main theorem

First consider an ASD connection over $\mathbb{R}^4$ with finite energy:

$$\int_{\mathbb{R}^4} |F_A|^2 d\mu < \infty$$

Due to conformal invariance, it can be regarded as an ASD connection over $S^4 - \{\infty\}$. According to the removable singularities theorem of Uhlenbeck(see [9]), this connection can be extended smoothly to S^4. In particular there is an integer invariant:

$$\kappa(A) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} |F_A|^2 d\mu$$

It also makes sense to talk about the fiber E_∞ of the bundle over S^4.

We denote by V the standard $\mathbb{R}^4$. The ASD equation can be written as:

$$[\nabla_1, \nabla_2] + [\nabla_3, \nabla_4] = 0$$

$$[\nabla_1, \nabla_3] + [\nabla_4, \nabla_2] = 0$$

$$[\nabla_1, \nabla_4] + [\nabla_2, \nabla_3] = 0$$

We will denote by I, J, K the standard basis for $\bigwedge^+$, as in the above form of ASD equation.

The ADHM construction gives a one-to-one correspondence between the gauge equivalence class of ASD connections for gauge group $SU(n)$ and a system of finite-dimensional algebraic data. We now describe them.

Data:

1. A k-dimensional Hermitian vector space $\mathscr{H}$;
2. An n-dimensional Hermitian vector space E_∞ and a determinant form in $\bigwedge^n E_\infty$;
3. Self-adjoint linear maps $T_i : \mathscr{H} \to \mathscr{H}, i = 1, 2, 3, 4$, or viewed as a linear map $T \in V^* \otimes \mathrm{Hom}(\mathscr{H}, \mathscr{H})$;

4. A linear map $P : E_\infty \to \mathscr{H} \otimes S^+$, where S^+ is the two-dimensional positive spin space of V described above.

Given such a system and a point $x \in \mathbb{R}^4$, we define a linear map:

$$R_x : \mathscr{H} \otimes S^- \oplus E_\infty \to \mathscr{H} \otimes S^+$$

by

$$R_x = (\sum_{i=1}^{4}(T_i - x_i 1) \otimes \gamma(e_i)^*) \oplus P$$

The product PP^* lies in End($\mathscr{H} \otimes S^+$) =End($\mathscr{H}$)⊗End(S^+). The space End(S^+) contains a direct summand $\mathfrak{su}(S^+)$, which is isomorphic to $\bigwedge^+$ via the map ρ in previous section. So there is a component of PP^*, which we denoted by $(PP^*)_{\bigwedge^+}$ in End$\mathscr{H} \otimes \bigwedge^+$. Using basis, there are components $PP_I^*, PP_J^*, PP_K^* \in$End($\mathscr{H}$).

Definition 3.2. *A system of ADHM data, for group $SU(n)$ and index k, is a system $(\mathscr{H}, E_\infty, T, P)$ as above which satisfies:*

1. *(The ADHM equations):*

$$[T_1, T_2] + [T_3, T_4] = PP_I^*$$

$$[T_1, T_3] + [T_4, T_2] = PP_J^*$$

$$[T_1, T_4] + [T_2, T_3] = PP_K^*$$

2. *(The non-degeneracy condition): for each $x \in \mathbb{R}^4$, the map R_x is surjective.*

The non-degeneracy condition means that the kernel of R_x defines a subbundle E of the trivial bundle $\mathscr{H} \otimes S^- \oplus E_\infty$. Then orthogonal projection gives a connection $A(T, P)$ on E. We have the following fact whose proof we omit here:

Proposition 3.3. *For any system of ADHM data (T, P), of index k, the connection $A(T, P)$ is ASD, of finite energy and $\kappa(A(T, P)) = k$.*

Now we fix model space $\mathscr{H} = \mathbb{C}^k$ and $E_\infty = \mathbb{C}^n$ so that all maps T_i, P becomes matrices. In this case, two systems (T, P) and (T', P') are said to be equivalent if there exist $v \in U(k)$ and $u \in SU(n)$ such that:

$$T_i' = vT_iv^{-1}, P' = vPu^{-1}$$

Then we can state our main theorem:

Theorem 3.2 (ADHM construction)**.** *The assignment $(T, P) \to A(T, P)$ sets up a one-to-one correspondence between (a) the equivalence classes of ADHM data for group $SU(n)$ and index k and (b) gauge equivalence classes of finite energy, ASD $SU(n)$-connection A over $\mathbb{R}^4$ with $\kappa(A) = k$.*

3.2.3 Complex of an ADHM system

In this part, we relate ADHM equations to complexes.

First we explain a canonical spin structure on a two-dimensional Hermitian vector space U with a determinant form $\theta \in \bigwedge^2 U$. We consider the sequence of maps:

$$0 \longrightarrow \mathbb{C} \xrightarrow{\delta_u} U \xrightarrow{\delta_u} \wedge^2 U \longrightarrow 0$$

where δ_u is the wedge product with u. This is exact for all $u \neq 0$. Since we fix a basis of $\bigwedge^2 U$, we can identify it with $\mathbb{C}$. Now let $S^+ = \mathbb{C} \oplus \mathbb{C}$ and $S^- = U$, we define $\gamma_u : S^+ \to S^-$ be the linear map $\gamma_u = \delta_u + \delta_u^*$. On S^+ we allow that $\langle 1 \rangle$ and $\langle \theta \rangle$ be the orthogonal basis. Then γ sets up a correspondence between U and $\mathrm{Hom}_J(S^+, S^-)$, hence defines a spin structure on U when it is regarded as a four-dimentional Euclidean space.

We write U for the base space when it is endowed with a complex structure and fix a basis $\theta \in \bigwedge^2 U$. Note that we use I to denote a basis element of $\bigwedge^+$ and also a complex structure. This is because a complex structure precisely corresponds to a choice of unit vector in $\bigwedge^+$.

Given a choice of complex structure, we introduce new variables:

$$\tau_1 = T_1 + iT_2, \tau_2 = T_3 + iT_4$$

Also, the complex structure decomposes S^+ into two pieces, thun we can write P as:

$$P = \pi^* \otimes \langle 1 \rangle + \sigma \otimes \langle \theta \rangle$$

where $\pi : \mathscr{H} \to E_\infty$ and $\sigma : E_\infty \to \mathscr{H}$. In terms of this new variables, the ADHM equations become:

1. $[\tau_1, \tau_2] + \sigma\pi = 0$
2. $[\tau_1, \tau_1^*] + [\tau_2, \tau_2^*] + \sigma\sigma^* - \pi^*\pi = 0$

Now we construct a sequence of maps:

$$\mathscr{H} \xrightarrow{\alpha} \mathscr{H} \otimes U \oplus E_\infty \xrightarrow{\beta} \mathscr{H}$$

where α, β in standard complex coordinates on $U = \mathbb{C}^2$ are given by:

$$\alpha = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \pi \end{pmatrix}, \beta = \begin{pmatrix} -\tau_2 & \tau_1 & \sigma \end{pmatrix}$$

Then the composite $\beta \circ \alpha$ is just the endomorphism $[\tau_1, \tau_2] + \sigma\pi$. Hence the first equation is just to say this defines a complex. More generally, replace τ_j by $\tau_j - z_j$, which doesn't affect the first equation, we obtain a family of complexes:

$$\mathscr{H} \xrightarrow{\alpha_x} \mathscr{H} \otimes U \oplus E_\infty \xrightarrow{\beta_x} \mathscr{H}$$

Then we have an identification $R_x = \alpha_x^* \oplus \beta_x$ when $\mathscr{H} \otimes U \oplus E_\infty$ is idenfitied with $\mathscr{H} \otimes S^- \oplus E_\infty$. This is just the identification of γ_u with $\delta_u \oplus \delta_u^*$ from the second equation. Thus we have:

Proposition 3.4. *A system (T, P) satisfies the ADHM euqations if and only if for each choice of complex structure on V, the maps α, β satisfies $\beta \circ \alpha = 0$.*

Finally, the non-degeneracy condition means that α_x are injective and β_x are surjective.

3.2.4 Construction in the opposite direction

In the above we constructed an ASD connection from a system of ADHM data. Now we do the construction in the opposite side. Starting from an ASD connection A over $\mathbb{R}^4$, Uhlenbeck's removable sigularities theorem gives us the fiber E_∞. So it remains to construct $\mathscr{H} = \mathscr{H}_A$ and maps P_A, T_A.

Consider the coupled Dirac operators $D_A : \Gamma(E \otimes_{\mathbb{C}} S^+) \to \Gamma(E \otimes_{\mathbb{C}} S^-)$. We know that over S^4 it is a Fredholm operator. We denote the kernel of D_A^* by $\mathscr{H}_A$. Transform to $\mathbb{R}^4$, we can represent an element of $\mathscr{H}_A$ as an E-valued spinor field ψ over $\mathbb{R}^4$ with decaying:

$$|\psi(x)| = O(|x|^{-3})$$

satisfying the differential equation $D_A^*\psi = 0$. Conversely, it can be shown that this is the only solutions of $D_A^*\psi = 0$ which are $O(|x|^{-1})$. Thus we can assume $\psi \in L^2$ and view $\mathscr{H}_A$ as the space of L^2 harmonic spinors. There is a normalized inner product on $\mathscr{H}_A$:

$$\langle \psi, \phi \rangle = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} (\psi, \phi) d\mu$$

On the other hand, returing to S^4, we have an evaluation map at the point ∞:

$$(ev) : \mathscr{H}_A \to [S^-]_\infty \otimes E_\infty$$

But there is a natural orientation-reversing isometry between the tangent spaces of S^4 at 0 and ∞, so we identify $[S^-]_\infty$ with the positive spin bundle S^+ of $\mathbb{R}^4$. Then (ev) maps to $S^+ \otimes E_\infty$. Now we can define the complex linear map $P_A : E_\infty \to \mathscr{H}_A \otimes S^+$, derived from the adjoint of the evaluation map and the skew isomorphism between S^+ and its dual.

Finally, we define T_i from the above construction. It is characterized by:

$$(T_i\psi, \phi) = \int_{\mathbb{R}^4} (x_i\psi, \phi) d\mu$$

We may view it is the composition of multiplication by coordinate functions with L^2 projection to $\mathscr{H}_A$.

The main content is the following:

Proposition 3.5. *For any finite energy ASD connection A on a bundle E over $\mathbb{R}^4$, the data $(\mathscr{H}_A, E_\infty, T_A, P_A)$ is a system of ADHM data and there are natural isomorphisms:*

$$\omega_x : Ker(R_x) \to E_x$$

giving a bundle map ω with $\omega^*(A) = A(T_A, P_A)$.

3.2.5 Sketch of the proof

Here we sketch the proof of the theorem. The main tool is the spectral sequence of a double complex. For a double complex $(\mathscr{C}^{**}, \delta_1, \delta_2)$ where $\delta_1 : \mathscr{C}^{p,q} \to \mathscr{C}^{p+1,q}$ and $\delta_2 : \mathscr{C}^{p,q} \to \mathscr{C}^{p,q+1}$ are homomorphisms, by the first spectral sequence $E_r^{p,q}$ we mean the one such that $E_1^{p,q} = H^p(\mathscr{C}^{*,q}, \delta_1)$. The another is called the second spectral sequence denoted by $\tilde{E}_r^{p,q}$.

Step1:

First we prove proposition 3.5, which will give one side of correspondence. We start with an instanton A. Given a complex structure I, we will first construct a double complex as well as maps α, β, τ, σ and π. We prove that (1) there is a natural holomorphic isomorphism $\omega : W = \frac{Ker(\beta)}{Im(\alpha)} \to E$; (2) these constructions satisfy algebraic condition in section 3.2.3.

Fix a complex structure on $\mathbb{R}^4$ and makes it into a complex space U. Consider the following double complex $\mathscr{A}^{p,q}$: they are subspaces of $\Omega^{0,p}(E) \otimes \wedge^q(U^*)$, satisfying the growth condition $\psi = O'(|x|^{-(p-q+2)})$ for $\psi \in \mathscr{A}^{p,q}$. Here $s = O'(|x|^{-m})$ means that $|\nabla^{(l)} s| = O(|x|^{-(l+m)})$ as $x \to \infty$ for any $l \geq 0$. The differentials in the complex are as follows: we take the Dolbeault $\bar{\partial}_A$ complex tensored with the fixed vector space $\wedge^q(U^*)$ in the horizontal directions; and take the Koszul multiplication defined on the $\wedge^*(U^*)$-term in the vertical directions. It can be written as:

$$\begin{array}{ccccc}
O'(|x|^0) & \xrightarrow{\bar{\partial}} & O'(|x|^{-1}) & \xrightarrow{\bar{\partial}} & O'(|x|^{-2}) \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
O'(|x|^{-1}) & \xrightarrow{\bar{\partial}} & O'(|x|^{-2}) & \xrightarrow{\bar{\partial}} & O'(|x|^{-3}) \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
O'(|x|^{-2}) & \xrightarrow{\bar{\partial}} & O'(|x|^{-3}) & \xrightarrow{\bar{\partial}} & O'(|x|^{-4})
\end{array}$$

And it can be easily deduced from the second sequence $\tilde{E}$ that:

Proposition 3.6. *The total cohomology of the complex is isomorphic to E_0 in dimension two, and otherwise zero.*

Next we turn to the first sequence to obtain the description of fiber E_0. Laborious analysis shows that:

Proposition 3.7. *The $E_1^{*,*}$ diagram of the double complex is:*

$$\begin{array}{c|ccc}
 & Ker(\sigma) \subset E_\infty & \mathscr{H}_A/Im(\sigma) & 0 \\
 & 0 & \mathscr{H}_A \otimes U & 0 \\
 & 0 & \mathscr{H}_A & 0 \\
\hline
 & & &
\end{array}$$

where $\sigma : E_\infty \to \mathscr{H}_A$ is defined by $\sigma(e) = \bar{\partial}_A(s_e)$.

Choose complex coordinates z_1, z_2 on U. Let $d_1 : \mathscr{H}_A \to \mathscr{H}_A \otimes U = \mathscr{H}_A \oplus \mathscr{H}_A$ be the map in the $E_1^{*,*}$ diagram. Then we define $\tau_j : \mathscr{H}_A \to \mathscr{H}_A$ be the components of d_1. Then we can check that the other differential $d_1 : \mathscr{H}_A \oplus \mathscr{H}_A \to \mathscr{H}_A / Im(\sigma)$ is the reduction mod $Im(\sigma)$ of the map:

$$\tilde{d}_1 = (-\tau_1, \tau_2) : \mathscr{H}_A \oplus \mathscr{H}_A \to \mathscr{H}_A$$

Following this, the only remaining differential is:

$$d_2 : \{Ker(d_1) \subset \mathscr{H}_A\} \to \{Ker\sigma \subset E_\infty\}$$

It can be extended to a map $\pi : \mathscr{H}_A \to E_\infty$ by following lemma:

lemma 3.3. *If f is a bounded section of E with $|\bar{\partial}_A f| = O'(|x|^{-2})$, then f extends to a limit in E_∞ as $x \to \infty$.*

It can be checked that $[\tau_1, \tau_2] = \sigma\pi$.

The spectral sequence gives an exact sequence:

$$0 \longrightarrow E_3^{0,2} \longrightarrow H \longrightarrow E_3^{1,1} \longrightarrow 0$$

where H is total cohomology of the double complex, which is known to be E_0. We write this sequence as:

$$0 \longrightarrow \frac{Ker(\sigma)}{Im(\pi)|_{Ker(a)}} \longrightarrow E_0 \longrightarrow \frac{b^{-1}(Im(\sigma))}{Im(a)} \longrightarrow 0$$

where $a = (\tau_1, \tau_2) : \mathscr{H}_A \to \mathscr{H}_A \oplus \mathscr{H}_A$ and $b = (-\tau_2, \tau_1) : \mathscr{H}_A \oplus \mathscr{H}_A \to \mathscr{H}_A$.

We can then define:

$$\alpha = (a, \pi) : \mathscr{H}_A \to \mathscr{H}_A \oplus \mathscr{H}_A \oplus E_\infty$$

$$\beta = (b, \sigma) : \mathscr{H}_A \oplus \mathscr{H}_A \oplus E_\infty \to \mathscr{H}_A$$

Since $[\tau_1, \tau_2] = \sigma\pi$, we know that $\beta \circ \alpha = 0$. Define $W = \frac{Ker(\beta)}{Im(\alpha)}$, then we have a natural exact sequence:

$$0 \longrightarrow \frac{Ker(\sigma)}{Im(\pi)|_{Ker(a)}} \longrightarrow W \longrightarrow \frac{b^{-1}(Im(\sigma))}{Im(a)} \longrightarrow 0$$

Finally, we see there is a natural map $\omega : W \to E_0$. This is defined as follows: a triple (ψ_1, ψ_2, e) represents an element of $Ker(\beta)$ if and only if there is a section f converging to e at infinity such that $\bar{\partial}_A f = z_1\psi_2 - z_2\psi_1$. Then put $\omega(\psi_1, \psi_2, e) = f(0) \in E_0$. It can be checked that we have the following commutation diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{Ker(\sigma)}{Im(\pi)|_{Ker(a)}} & \longrightarrow & W & \longrightarrow & \frac{b^{-1}(Im(\sigma))}{Im(a)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \frac{Ker(\sigma)}{Im(\pi)|_{Ker(a)}} & \longrightarrow & E_0 & \longrightarrow & \frac{b^{-1}(Im(\sigma))}{Im(a)} & \longrightarrow & 0
\end{array}$$

Hence ω is an isomorphism. Similiarly, replace τ_j by $\tau_j - z_j$, we get a sequence of holomorphic bundle maps, which gives a holomorphic cohomology bundle $\mathscr{W}$. It can shown that $\omega : \mathscr{W} \to \mathscr{E}$ is actually a holomorphic bundle isomorphism.

Then we show that these holomorphic data (τ, π, σ) precisely relate to our data $(\mathscr{H}_A, E_\infty, T_A, P_A)$ via our construction in section 3.2.3. This shows that the data $(\mathscr{H}_A, E_\infty, T_A, P_A)$ is indeed an ADHM data.. Recall that we have an evaluation map $(ev) : \mathscr{H}_A \to E_\infty \otimes S^+$. Using basis, we can write this as:

$$(ev) = (ev)_1 \otimes \langle 1 \rangle + (ev)_\theta \otimes \langle \theta \rangle$$

for $(ev)_1, (ev)_\theta : \mathscr{H}_A \to E_\infty$. We have the following proposition:

Proposition 3.8. *The evaluation map is related to the maps σ and π. Actually, we have $(ev)_1^* = \sigma$ and $(ev)_\theta = \pi$.*

Now that since ω is a holomorphic isomorphism, we have $\omega^*(A) = A(T_A, P_A)$ due to the uniqueness of a Chern connection. Note that ω is independent of the choice of complex structure, we then finish this part of the proof.

Step2:

Then we prove that starting with an ADHM data $(\mathscr{H}, E_\infty, T, P)$, we recover the same matrix data from $A = A(T, P)$. Consider the following double complex:

$$\begin{array}{ccccc}
O'(|x|^{-4}) & \xrightarrow{\alpha} & O'(|x|^{-3}) & \xrightarrow{\beta} & O'(|x|^{-2}) \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow \\
O'(|x|^{-3}) & \xrightarrow{\alpha} & O'(|x|^{-2}) & \xrightarrow{\beta} & O'(|x|^{-1}) \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow \\
O'(|x|^{-2}) & \xrightarrow{\alpha} & O'(|x|^{-1}) & \xrightarrow{\beta} & O'(|x|^{0})
\end{array}$$

Here the entries are the forms with values in the trivial bundle $\mathscr{H}$, $\mathscr{H} \oplus \mathscr{H} \oplus E_\infty$ and $\mathscr{H}$ respectively, satisfying the stated growth condition. The cohomology of the rows yields the E-valued forms, and taking the vertical cohomology we get the $E_2^{*,*}$ diagram:

$$\begin{array}{|ccc}
0 & 0 & 0 \\
0 & \mathscr{H}_A & 0 \\
0 & 0 & 0 \\
\hline
\end{array}$$

On the other hand, since all the original bundle are trivial, the only cohomology in the columns is:

$$\begin{array}{|ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mathscr{H} \\
\hline
\end{array}$$

This yields the desired isomorphism between $\mathscr{H}$ and $\mathscr{H}_A$. We can verify that under this isomorphism, T_i and P indeed correspond to the multiplication and evaluation maps. This ends the proof of the theorem.

4 Application to four-manifolds

This section is to discuss applications of gauge theory to the study of four-dimensional differential topology. We will first explain the concept of intersection forms, which are central tools in the study of manifold topology; then we review results proved by purely topological methods; finally we talk about Donaldson's diagnoalizable theorem and sketch its proof.

4.1 Intersection form

Guiding problems in differential topology study are mainly the following two:

1. geography problem: find all simply-connected manifolds which admit a smooth structure
2. botany problem: find all exotic smooth structures on a given smooth manifold

A useful invariant is the intersection form, for which we are going to explain.

4.1.1 Definition

For a closed oriented manifold M^n, we can define the cup product pairing as follows:

$$H^k(M;\mathbb{Z}) \otimes H^{n-k}(M;\mathbb{Z}) \to \mathbb{Z}, (\alpha, \beta) \mapsto (\alpha \cup \beta)[M]$$

Due to Poincaré duality, the pairing is nonsingular when torsion is factored out.

We focus on the case that M^4 is an oriented simply-connected closed manifold. Therefore the first and third homology group vanish and $H^2(M;\mathbb{Z}) \cong \text{Hom}(H_2(M), \mathbb{Z})$ is free due to universal coefficient theorem. The intersection form of M is thus defined to be the nonsingular pairing:

$$Q : H^2(M;\mathbb{Z}) \otimes H^2(M;\mathbb{Z}) \to \mathbb{Z}, (\alpha, \beta) \mapsto (\alpha \cup \beta)[M]$$

Note that Q is a symmetric bilinear form.

The dual of intersection form coincides with the intersection product for homology. For detailed information, see [1]. Geometric meaning is as follows. Assume we have two oriented embedded smooth surfaces A and B of M which intersect transversally, hence their intersection $A \cap B$ being a finite subset of M. Let $\alpha, \beta \in H^2(M;\mathbb{Z})$ are respectively Poincaré dual to the fundamental class of A and B in $H_2(M)$, then it can be shown that $Q(\alpha, \beta)$ is equal to the total number of intersection points counted with signs. Note that the counting is also symmetric with respect to A and B, consistent with Q.

Therefore the computation of intersection form can be reduced to calculate signed total number of intersection points via appropriate choice of submanifold. In our case, any homology class can be realized by embedded smooth surface due to a more general result:

Theorem 4.1 (Thom)**.** *If M^n is an oriented close manifold, then any homology class in $H_{n-1}(M)$ and $H_{n-2}(M)$ can be represented by the fundamental class of a smoothly embedded oriented submanifold, possibly disconnected in the case of codimension 1.*

For a proof, see [1]. For a easy proof concerning the H_{n-2} assertion, see the appendix in [8].

4.1.2 Classical invariants v.s intersection form

We are going to show that some classical invariants are the same in two manifolds with equivalent intersection forms. Therefore they reflects no more information than intersection form. These invariants are Stiefel-Whitney class, Pontryagin class and Euler class.

First we consider the Stiefel-Whitney class. Since M is simply-connected, $H^1(M;\mathbb{Z}_2)$ and $H^3(M;\mathbb{Z}_2)$ must vanish. Therefore w_1, w_3 vanish and the only possibly non-trivial class is w_2[1]. By Wu's formula(see [3]), $w_2 = v_2$ where v_2 is the second Wu class characterized by

$$\langle v_k \cup x, [M]\rangle = \langle Sq^k(x), [M]\rangle$$

Hence:

$$\langle w_2 \cup x, [M]\rangle = \langle x \cup x, [M]\rangle.$$

The following result from algebra tells us w_2 can be obtained from the intersection form:

Proposition 4.1. *Every symmetric bilinear unimodular form Q over $\mathbb{Z}^r$ admits a characteristic element c, i.e. $Q(c,x) \equiv Q(x,x) (mod\ 2)$.*

Hence w_2 is the module 2 reduction of a characteristic element.

Recall Q is called *even* if $Q(x,x)$ is even for every x; otherwise it is called odd. Therefore the previous formula shows that:

Proposition 4.2. $w_2 = 0$ *if and only if Q is even.*

For an oriented vector bundle E over M, there exists a spin structure if and only if $w_2(E) = 0$, and is unique if $H^1(M;\mathbb{Z}_2) = 0$. Thus we have proved:

Theorem 4.2. *For an oriented simply-connected 4-manifold M, it admits a spin structure if and only if the intersection form is even. If so, it is unique.*

Next we consider the Pontryagin class. The signature τ is given by Hirzebruch signature theorem in dimension 4 as follows:

$$p_1 = 3\tau = 3(b^+ - b^-)$$

Finally Euler class is given obviously by:

$$e = \Sigma(-1)^i b_i = b_2 + 2$$

[1]In fact for any oriented four-manifold, w_1 and w_3 vanish. For a proof, see [3]

4.1.3 Examples

We compute the intersection form of some basic examples.

(i) Standard sphere S^4. Since $H_2(S^4) = 0$, all intersection numbers vanish.

(ii) Complex projective space $\mathbb{C}P^3$. Since $\mathbb{C}P^1$ generates $H_2(M)$, we choose two embedded P^1 intersecting transversally, say $P^1 = \{(z_0, z_1, 0) : |z_0|^2 + |z_1|^2 \neq 0\}$, $\widetilde{P^1} = \{(z_0, 0, z_2) : |z_0|^2 + |z_2|^2 \neq 0\}$. They intersect at one positive point, hence the intersection matrix is (1). We denote the complex projective space with opposite orientation by $\overline{\mathbb{C}P^2}$, then its intersection matrix is (-1)

(iii) $S^2 \times S^2$. By Künneth formula, $S^2 \times \{p\}$ and $\{q\} \times S^2$ generates $H_2(M)$. Consider $S^2 \times \{p\}$ and $S^2 \times \{p\prime\}$ to compute the diagonal entries,$S^2 \times \{p\}$ and $\{q\} \times S^2$ to compute the off-diagonal entries. The intersection matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(iv) $S^2 \times S^2$ can be viewed as compactifying the fiber of the trivial bundle $S \times \mathbb{C}$. Note that complex line bundles are classified by first Chern class(see). Starting with any complex line bundle L with Chern class $c_1(L) = d$, we compactifying each fiber to S^2 and thus obtain a fiber bundle M_d.

It's easy to see $Q_{11} = 0, Q_{12} = Q_{21} = 1$. For the self-intersection number of zero section, we claim that there exists a smooth section transversal to zero section and hence has finitely many zero points(see [1]). Since the the first Chern class of line bundle is also the Euler class of L, the fact that $Q_{22} = d$ follows from that the Euler class of a vector bundle over compact manifold is Poincaré dual to the zeroes of a section that vanishes at finitely many points. Therefore the intersection matrix is $\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$.

There are in fact exactly two diffeomorphism types of M_d, i.e. M_d is diffeomorphic to M_0 is d is even and diffeomorphic to M_1 is d is odd. This is because S^2 is covered by upper hemisphere and lower hemisphere. Both are contractble and hence the fiber bundles over them are trivial. Therefore M_d is determined by the transition function, i.e. determined by the homotopy class of the map $g : S^1 \to Diff(S^2)$. The conclusion follows from the fact that $Diff(S^2)$ is homotopy equivalent to $SO(3)$ and $\pi_1(SO(3)) = \mathbb{Z}_2$.

We point out here that M_1 is actually $\mathbb{C}P_2 \# \overline{\mathbb{C}P^2}$.

(v) $X \# Y$. Since $H_2(X \# Y) \cong H_2(X) \oplus H_2(Y)$, the intersection form $Q(X \# Y) = Q(X) \oplus Q(Y)$. For example, $l\mathbb{C}P_2 \# m\overline{\mathbb{C}P^2}$ has intersection form $l\,(1) \oplus m\,(-1)$.

(vi) Hypersurfaces in $\mathbb{C}P^3$. Consider a smooth hypersurface S_d of degree d in $\mathbb{C}P^3$, for example $z_0^d + z_1^d + z_2^d + z_3^d = 0$. Lefschetz hyperplane theorem asserts that the homotopy groups of a hypersurface of complex dimension n in projective space agree with those of the ambient space up to dimension $n-1$, so S_d is simply-connected. For instance, S_4 has intersection matrix as $3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\oplus 2(-E_8)$ where E_8 will be given in next part.

4.2 Algebraic classification of unimodular forms

We have seen some examples of intersection form above. Now we list some algebraic results about unimodular forms. For their proof, see [6].

First we point out some results about characteristic element.

Proposition 4.3. *Let c be a characteristic element of Q. Then $Q(c,c) \equiv$ signature (mod 8).*

If Q is even, 0 is a characteritic element of Q. Hence:

Corollary 4.1. *If Q is even, then signature $\equiv 0$(mod 8).*

For indefinite unimodular forms, we have the following classification theorem:

Theorem 4.3 (Hasse-Minkowski)**.** *Any odd indefinite form is uniquely equivalent to $l\,(1) \oplus m\,(-1)$; any even indefinite form is uniquely equivalent to $l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8$ where*

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & -1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

However, the classification of definite forms is still unsolved. And there are numerous definite forms equivalence classes, say existing at least 10^7 even definite forms of rank 32.

4.3 Topological classification of four-manifolds

4.3.1 Homotopy type

We say two manifolds are oriented homotopy equivalent if they are homopoty equivalent and the equivalence map preserves the fundamental class. Clearly intersection forms are invariant under oriented homotopy equivalence. The converse holds due to Milnor:

Theorem 4.4 (Milnor)**.** *Oriented closed simply-connected 4-manifolds are oriented homotopy equivalent if and only if their intersection forms are equivalent.*

This is proved by using a more general theorem of Whitehead.

By Thom's cobordism theorem, two closed manifolds with the same Stiefel-Whitney numbers can be cobordant. Wall provided a stronger result, i.e. such two 4-manifolds X and Y can be h-cobordant, which means there exists a simply-connected compact manifold W^5 with boundary $X \bigsqcup \overline{Y}$, and the inclusions of X and Y in W are both homotopy equivalences. His proof involves "surgery" to kill the fundamental group of W and to rearrange and cancell some critical points of a given Morse function.

Theorem 4.5 (Wall)**.** *Oriented closed simply-connected 4-manifolds have equivalent intersection forms if and only if they are h-cobordant. If so, there exists $k \geq 0$ s.t. $X \# k(S^2 \times S^2)$ is diffeomorphic to $Y \# k(S^2 \times S^2)$.*

However, what we really want is whether two manifolds are differmorphic. Note that in dimension 5 or higher, we have Smale's h-cobordism theorem:

Theorem 4.6 (Smale)**.** *If $W^n (n \geq 6)$ is a simply-connected compact manifold which is a h-cobordism of X and Y. Then there exists a Morse function without critical points in W, i.e. W is a product and X, Y are diffeomorphic.*

If h-cobordism theorem held in dimension 4, then intersection forms would truly determine the diffeomorphic type of 4-manifolds. But the proof of h-cobordism theorem fails in dimension 4, and Donaldson's later result shows that in fact it doesn't hold in dimension 4. Therefore the differential topology of 4-manifolds is more subtle.

4.3.2 Realization of unimodular forms

Not all forms can be realized as intersection form of a closed simply-connected 4-manifold. A constraint is provided by a deep theorem of Rohlin:

Theorem 4.7 (Rohlin)**.** *The signature of a smooth, compact, spin 4-manifold is divisible by 16.*

Therefore E_8 cannot be realized since it is even, and hence the possible manifold must be spin. But its signature is 8.

However we have the following alternating result:

Theorem 4.8. *For any unimodular form Q, there is a simply-connected 4-manifold with boundary, having intersection form Q and boundary a homology S^3.*

4.3.3 Freedman's result

Freedman solved this topological problem completely.

Theorem 4.9 (Freedman)**.** *For any integral symmetric unimodular form Q, there is a closed simply-connected topological 4-manifold that has Q as its intersection form. More precisely,*

1. *If Q is even, there is exactly one such manifold;*
2. *If Q is odd, there are exactly two such manifolds, at least one of which doesn't admit any smooth structures.*

He also used another invariant to clarify the case when Q is odd. Thus his classification is of full completion. In particular, two smooth simply-connected 4-manifolds with isomorphic intersection forms are homeomorphic.

4.4 Donaldson's diagonalizable theorem

However, in smooth realm, Donaldson proved the following surprising result:

Theorem 4.10 (Donaldson)**.** *The only definite forms that can be realized by smooth 4-manifolds are just $\oplus m\,(1)$ and $\oplus m\,(-1)$.*

Thus, none of exotic definite forms can be realized by smooth 4-manifolds. Several results can be deduced from Donaldson's theorem:

1. Any smooth simply-connected 4-manifold is homeomorphic to $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ or $\# \pm m\mathcal{M}_8\#n(S^2 \times S^2)$ where $\mathcal{M}_8$ is the topological manifold whose intersection form is E_8.

2. A large number of topological 4-manifolds cannot admit a smooth structure

3. Existence of fake $\mathbb{R}^4$

We sketch how Donaldson proved this remarkable result via instanton moduli space. He considered the $SU(2)$-bundle with Pontrjagin index -1 over M, and study the topology of moduli space $\mathcal{M}$.

It turns out that $\mathcal{M}$ is a smooth 5-manifold with m singularities, and around these singularities are like a cone in $\mathbb{C}P^2$. Here m is half the solutions to $Q(\alpha, \alpha) = 1$. Moreover, $\mathcal{M}$ contains a collar of M such that $\overline{\mathcal{M}} = \mathcal{M} \cup M$ is a compact oriented smooth manifold with boundary. Therefore, **M is oriented cobordant to m disjoint complex projective spaces**, i.e.$\pm\mathbb{C}P^2 \amalg ... \amalg \pm\mathbb{C}P^2$.

Final attack comes from a simple algebraic lemma:

lemma 4.1. *Let Q be a positive definite symmetric unimodular form with rank r, m is defined as before. Then $m \leq r$, where the equality holds if and only if Q is diagonalizable over $\mathbb{Z}$.*

Now we can proof Donaldson's result. Since the signature of intersection form is an oriented cobordism invariant, it follows that:

$$r = \sigma \leq m\sigma(\mathbb{C}P^2) = m$$

But from lemma, $m \leq r$. Hence the equality holds and Q is diagonalizable. **Q.E.D.**

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